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*Research Article*

# Attractors of Compactly Generated Semigroups of Regular Polynomial Mappings

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We investigate the metric space of pluriregular sets as well as the contractions on that space induced by infinite compact families of proper polynomial mappings of several complex variables. The topological semigroups generated by such families, with composition as the semigroup operation, lead to the constructions of a variety of Julia-type pluriregular sets. The generating families can also be viewed as infinite iterated function systems with compact attractors. We show that such attractors can be approximated both deterministically and probabilistically in a manner of the classic chaos game.

## 1. Introduction

In the recent paper [1] it was shown, as a part of the investigation of the space of pluriregular sets, that it is possible to approximate composite Julia sets generated by finite families of proper polynomial mappings in $\mathbb{C}^N$ in a probabilistic manner. This can be done in the spirit of the theory of iterated function systems (IFSs) and the so-called chaos game. The aim of this paper is to prove similar results in the case of infinite compact families of polynomial mappings. Inevitably, the topological and probabilistic aspects get more complicated than those in the finite case. The main motivation for this study is the wish to gain a better understanding of the metric space $\mathscr{R}$ of compact, pluriregular, and polynomially convex subsets of $\mathbb{C}^N$. This, however, requires a very careful analysis of different types of Julia-like sets arising naturally in this context. This variety of Julia sets is easier to grasp, if one looks at them as corresponding to the topological semigroup generated by infinite compact families of proper polynomial mappings. This is consistent with the point of view adopted by a number of researchers in one complex variable (see [2] and, e.g., the work of Stankewitz and Sumi [3–5]).

As a visual hint of the additional complexity that infinite families bring about, we can consider what happens in the complex plane when instead of inspecting the filled-in Julia set of a single polynomial, in this case $p_c(z) = z^2 + c$ with $c = c_0$, we examine the filled-in Julia set generated by the compact infinite family of polynomials $\{p_c : c \in K\}$, where $K$ is a closed-square centered at $c_0$. In the following pictures, $c_0 = 0.3 + 0.5i$ and $K = c_0 + [-0.1,0.1] + i[-0.1,0.1]$. Figure 1 shows the autonomous Julia sets of the polynomials $p_c$, one with $c = c_0$ and the other eleven with $c$ selected at random from $K$ according to the uniform probability distribution.

Bearing in mind that this is just a tiny selection of Julia sets of the simplest (autonomous) type, one can appreciate the infinite variety of Julia sets (autonomous or not) that can be obtained by using just this family of simple quadratic polynomials. The union of all these sets would constitute the composite nonautonomous Julia set corresponding to all combinations of $c \in K$. An approximate outline of this set is depicted in Figure 2, to the right of the filled-in Julia set for $p_{c_0}$ included for comparison. All these sets were plotted with the help of measuring the escape time of the orbits of the points under the iteration process. The shades of grey mark how quickly the considered orbits

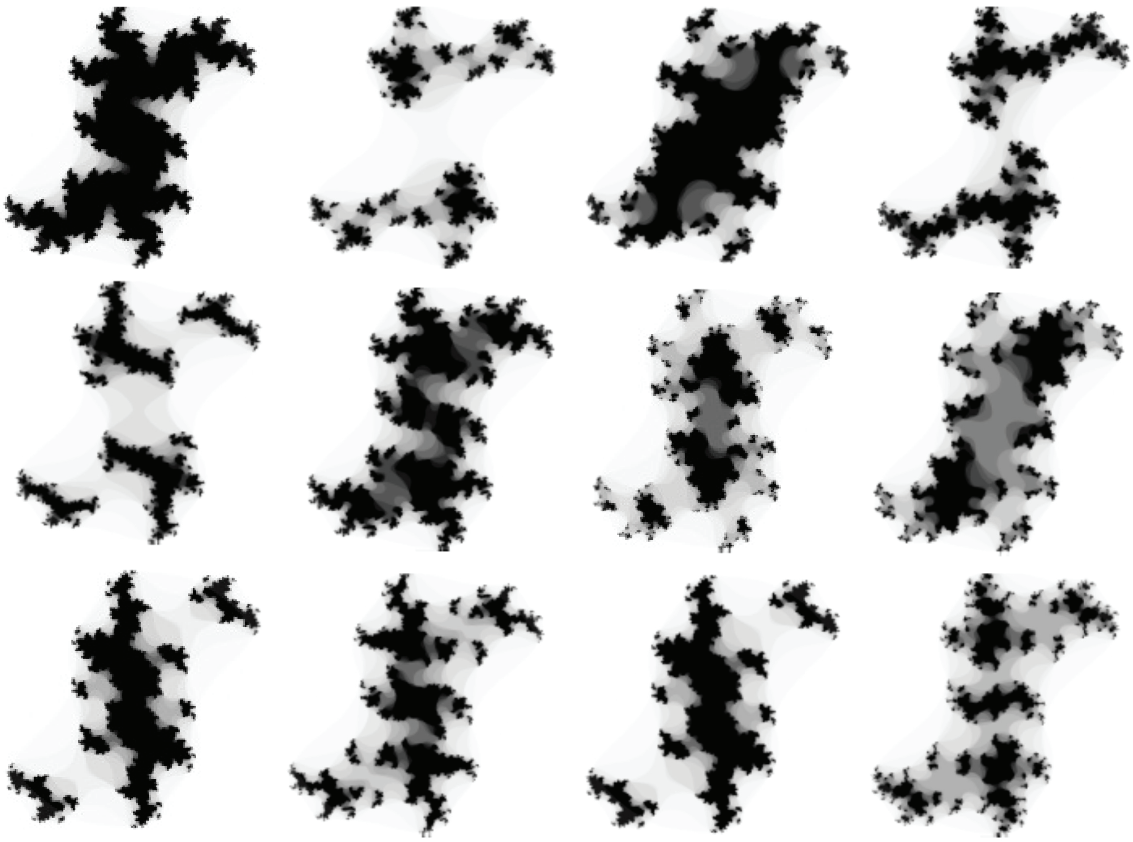

Figure 1

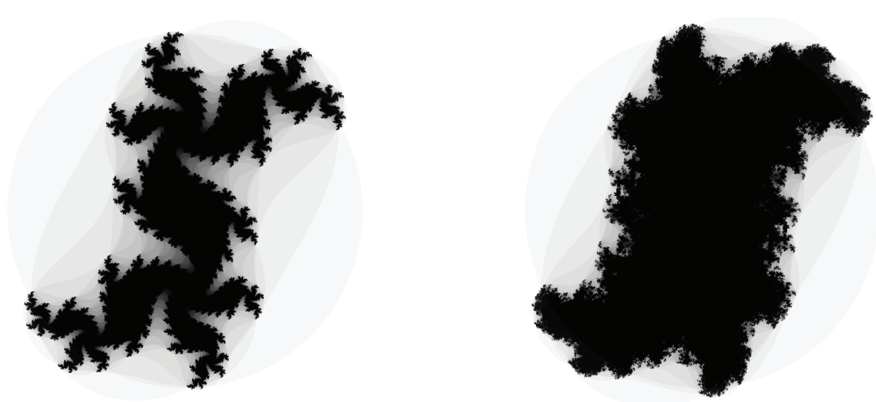

Figure 2

go beyond the radius of escape. Obviously, the situation gets more involved with more complicated polynomial mappings and in higher dimensions.

The paper is divided into seven sections including the introduction.

In Section 2, we take a closer look at the nature of convergence in the compact-open topology in the space of polynomial mappings in $\mathbb{C}^N$ and, in particular, at the link to the coefficients of such mappings and their compositions. We also recall the definition of regular polynomial mappings and the concept of radius of escape and its basic properties. Moreover, we propose to regard the topological semigroups generated by compact families of regular mappings, with the composition of mapping as the semigroup operation, as the principal objects that give rise to the composite Julia sets that we want to study.

In Section 3, we recall the definition of the pluricomplex Green function of a nonempty compact subset of $\mathbb{C}^N$ and the concept of pluriregularity. We also review the definition of the metric space $\mathscr{R}$ of all polynomially convex pluriregular compact sets in $\mathbb{C}^N$. The Julia sets we are studying in this paper reside precisely in that space. The space $\mathscr{R}$ is known to be complete (see [6]), and separable (see [1]) but not proper, in the sense that bounded closed sets do not have to be compact (see [1]). The topology of $\mathscr{R}$ still holds many unanswered questions. We discuss in some detail the intricacies of the closure operation in $\mathscr{R}$. Namely, given a subset $\mathscr{G}$ of a subset of $\mathscr{R}$, we compare the closure of the union in $\mathbb{C}^N$ of the sets which are elements of $\mathscr{G}$ with the union in $\mathbb{C}^N$ of the sets which are elements of the closure of $\mathscr{G}$ in $\mathscr{R}$. It turns out that equality between these sets requires additional assumptions.

In Section 4, we prove that if $P : \mathbb{C}^N \to \mathbb{C}^N$ is a regular polynomial mapping, then the contraction $A_P : \mathscr{R} \ni E \mapsto P^{-1}(E) \in \mathscr{R}$ is a similitude, which is also continuous when regarded as a function of two variables $(P, E) \mapsto A_P(E)$. We also show that if $\mathscr{F}$ is a compact family of regular polynomial mappings of a fixed degree and $\mathscr{K} \subset \mathscr{R}$ is compact, then $\mathscr{A}_{\mathscr{F}}(\mathscr{K}) = \bigcup_{P\in\mathscr{F}} A_P(\mathscr{K})$ is also compact. Furthermore, $\mathscr{A}_{\mathscr{F}} : \kappa(\mathscr{R}) \to \kappa(\mathscr{R})$ is a contraction whose fixed point $\mathscr{S}[\mathscr{F}]$ can be described as the atlas of the Julia sets generated by sequences from the topological semigroup generated by $\mathscr{F}$. We could also describe the set $\mathscr{S}[\mathscr{F}]$ as the attractor of the infinite iterated function system $\{A_P : P \in \mathscr{F}\}$.

Section 5 begins with restating the definitions of autonomous filled-in Julia set $J[P]$ generated by a single regular polynomial mapping $P$ and a nonautonomous filled-in Julia set $J[(P_n)_{n=1}^{\infty}]$ generated by a sequence $(P_n)_{n=1}^{\infty}$ of regular mappings. If the sequence comes from a compact family $\mathscr{F}$ of regular mappings with a fixed degree, then we show that $J[P_m \circ \ldots \circ P_1]$ converges to $J[(P_n)_{n=1}^{\infty}]$ as $m \to \infty$. Moreover, the speed of the convergence can be estimated in terms of the natural metric on $\mathscr{R}$. We also furnish the code space $\mathscr{F}^{\mathbb{N}}$ with a metric like the one used in the classical case when the family $\mathscr{F}$ is finite. We close this section by linking the attractor to other types of Julia sets. Namely, $\mathscr{S}[\mathscr{F}]$ consists of all sets $J[(P_n)_{n=1}^{\infty}]$ with $(P_n)_{n=1}^{\infty} \in \mathscr{F}^{\mathbb{N}}$ and the union of all sets constituting $\mathscr{S}[\mathscr{F}]$ is the partly filled-in composite Julia set $\mathbb{J}_{\mathrm{tr}}[\mathscr{F}]$ generated by $\mathscr{F}$, whereas the polynomially convex hull of $\mathbb{J}_{\mathrm{tr}}[\mathscr{F}]$ is the filled-in composite Julia set $\mathbb{J}[\mathscr{F}]$ generated by $\mathscr{F}$. We also include some comments to justify the use of semigroup terminology in this context.

The last two sections contain a counterpart of Theorem 2 in [1] in the case of a compact infinite family $\mathscr{F}$ of regular polynomial mappings of the same degree. Section 6 presents an extension of Theorems 2(a) and 2(b) from [1]. Essentially, we show how much the attractor $\mathscr{S}[\mathscr{F}]$ pulls iterations of sets from the surrounding space towards itself if the polynomial mappings used in the iteration process come from $\mathscr{F}$. In Section 7 we extend Theorem 2(c) from [1] to the case of compact infinite families $\mathscr{F}$ and we prove that the chaos game approximation of the partly filled-in composite Julia sets remains also valid in this case. We describe first the deterministic version based on disjunctive sequences and then the more familiar probabilistic version. Finally, we close the article with a few comments linking the mathematical context we have investigated to the study of invariant measures associated with general probabilistic approach to iteration function systems described in [7].

A few words about the notation used in this paper are in order. For any nonempty sets A and B, the symbol $\mathrm{B}^{\mathrm{A}}$ will denote the set of all functions from A to B. If $\mathscr{F}$ is a collection of nonempty subsets of a set $G$, the symbol $\bigcup\mathscr{F}$ will always denote $\bigcup_{F\in\mathscr{F}} F \subset G$. Let $(X, \mathfrak{d})$ be a metric space. The symbol $\kappa(X)$ will denote the set of all nonempty compact subsets of $X$; $B_{\mathfrak{d}}(a, r)$ will denote the open ball with center $a$ and radius $r$ whereas $\mathrm{dist}_{\mathfrak{d}}$, $\mathrm{diam}_{\mathfrak{d}}$, and $\chi_{\mathfrak{d}}$ will denote the distance of a point from a set, the diameter of a set, and the Hausdorff distance between two compact sets, respectively. The set $\{x \in X : \mathrm{dist}_{\mathfrak{d}}(x, E) \le \varepsilon\}$, where $\varepsilon > 0$, will be referred to as the $\varepsilon$-dilation of the set $E \subset X$. In the case of the Euclidean metric in $\mathbb{C}^N$, we will drop the subscript $\mathfrak{d}$. A norm symbol with a subscript will always denote the supremum norm. We will use the convention that $\mathbb{Z}_+$ stands for nonnegative integers and $\mathbb{N}$ for natural numbers (excluding zero). Other notational conventions will be described later as the need for them arises.

## 2. Semigroups of Regular Polynomial Mappings

If $d \in \mathbb{Z}_+$, then by $\mathscr{P}_d$ we denote the vector space of all polynomial mappings $P : \mathbb{C}^N \to \mathbb{C}^N$ of degree not greater than $d$. Since $\mathscr{P}_d$ is of finite dimension, all norms defined on it are equivalent. In particular, if $E \subset \mathbb{C}^N$ is compact and determining for polynomials (i.e., $E$ is not contained in the zero set of a nonconstant polynomial), then a natural choice is the supremum norm $\|P\|_E = \sup\{\|P(z)\| : z \in E\}$, where $P \in \mathscr{P}_d$, and $\mathbb{C}^N$ is endowed with the Euclidean norm. Another natural choice would be to transfer the norm from the Euclidean space of Taylor's coefficients using the natural isomorphism:

$$\mathbb{T}_d : \mathscr{P}_d \ni P \mapsto \left(\frac{D^\alpha P(0)}{\alpha!}\right)_{|\alpha|\le d} \in \left(\mathbb{C}^N\right)^{N_d} = \mathbb{C}^{N\cdot N_d}, \qquad (1)$$

where the multi-indices in $\mathbb{Z}_+^N$ are ordered according to the graded lexicographic order and

$$N_d = \binom{N+d}{d}. \qquad (2)$$

When $E(R)$ is the closed polydisc with the center at the origin and radius $R > 0$, then we can use Cauchy's estimates to establish a quantitative link between these two norms. For any $P \in \mathscr{P}_d$, if $P(z) = \sum_{|\alpha|\le d} z^\alpha p_\alpha$, with some $p_\alpha \in \mathbb{C}^N$, then we have the following:

$$\|P\|_{E(R)} \le \sum_{|\alpha|\le d} \|p_\alpha\| R^{|\alpha|} \le N_d \sqrt{N} \|P\|_{E(R)}. \qquad (3)$$

Consequently, the topology on $\mathscr{P}_d$ is the topology of uniform convergence of polynomial mappings on compact sets or, equivalently, the topology of convergence of the coefficients of polynomial mappings. To put it differently, it is the topology induced on $\mathscr{P}_d$ from the set $\mathscr{P}$ of all polynomial mappings $P : \mathbb{C}^N \to \mathbb{C}^N$ furnished with the compact-open topology, that is, the smallest topology containing all the sets of the form $N(K,U)=\{P\in\mathscr{P}:P(K)\subset U\}$, where $K\subset\mathbb{C}^N$ is compact and $U\subset\mathbb{C}^N$ is open. The following statement will be useful later on.

**Proposition 1.** *The composition mapping*

$$\mathscr{P}\times\cdots\times\mathscr{P}\ni(P_1,\dots,P_k)\mapsto P_k\circ\dots\circ P_1\in\mathscr{P} \tag{4}$$

*is continuous. In fact, if polynomial mappings are identified with their ordered sets of coefficients, then the mapping*

$$\mathscr{P}_{d_1}\times\cdots\times\mathscr{P}_{d_k}\ni(P_1,\dots,P_k)\mapsto P_k\circ\dots\circ P_1\in\mathscr{P}_{d_1\cdot\ldots\cdot d_k}, \tag{5}$$

*is a polynomial mapping between the respective spaces of coefficients.*

*Proof 1.* It suffices to consider $k=2$.

The first statement can be checked directly on the sets from the neighbourhood subbase $\{N(K,U)$: $K$—compact, $U$—open$\}$ of the topology of $\mathscr{P}$. If $Q\circ P\in N(K,U)$, then for some compact set $L\subset\mathbb{C}^N$, we can have the inclusions $P(K)\subset\operatorname{int}(L)$ and $Q(L)\subset U$. This means that $N(K,\operatorname{int}(L))\times N(L,U)$ is contained in the inverse image of $N(K,U)$ under the composition mapping, which completes the proof of continuity.

As for the second statement, in view of (1) and (3) it is enough to observe that the mapping

$$\begin{aligned}\mathbb{C}^{N\cdot N_{d_1}}\times\mathbb{C}^{N\cdot N_{d_2}}\ni\left(\mathbb{T}_{d_1}(P_1),\mathbb{T}_{d_2}(P_2)\right)\mapsto\mathbb{T}_{d_1d_2}(P_2\circ P_1)\\ \in\mathbb{C}^{N\cdot N_{d_1d_2}}\end{aligned} \tag{6}$$

is a polynomial.

If $P\in\mathscr{P}_d$, we will denote by $\widehat{P}$ the homogeneous component of $P$ of degree $d$. We say that $P\in\mathscr{P}_d$ is *regular* if $\widehat{P}^{-1}(0)=\{0\}$. The subset of all regular maps in $\mathscr{P}_d$, denoted by $\mathscr{P}_d^\star$, is an open subset of $\mathscr{P}_d$ (see Section 2 of [8]). Regular polynomial mappings are proper (cf. [9], Theorem 5.3.1) and so they are closed. As proper holomorphic mappings, they are also open and hence surjective (see [10], p. 301).

Throughout this paper, $\mathbb{B}_R$ will denote the closed Euclidean ball in $\mathbb{C}^N$ with center at the origin and radius $R>0$. If $P\in\mathscr{P}_d^\star$, then $P^{-1}(\mathbb{B}_R)=\overline{\operatorname{int}(P^{-1}(\mathbb{B}_R))}$.

In what follows, let $P^n$ denote the $n$th iterate of $P$, that is, the composition of $n$ copies of $P$. We call $R>0$ an *escape radius* for $P\in\mathscr{P}_d^\star$, if for every $z\in\mathbb{C}^N\backslash\mathbb{B}_R$, we have

$$\lim_{n\to\infty}\|P^n(z)\|=\infty. \tag{7}$$

Note that if $R>0$ is an escape radius for $P$, then all numbers bigger than $R$ are also escape radii for the same mapping.

In [11] (Lemma 1), it was proved that there exists a continuous function,

$$\mathscr{P}_d^\star\ni P\mapsto r(P)\in(0,\infty), \tag{8}$$

such that $r(P)$ (given by a constructive formula) is an escape radius for $P$. Another useful observation is that if $R\geq r(P)$, then $P^{-1}(\mathbb{B}_R)\subset\operatorname{int}\mathbb{B}_R$ (cf. [11], Lemma 1).

In our investigation, we will consider a nonempty compact subset $\mathscr{F}$ of $\mathscr{P}_d^\star$. It is worth mentioning that such a family is regular in the sense defined in [12]. Indeed, a subset of $\mathscr{P}_d$ is regular there if and only if it is relatively compact in $\mathscr{P}_d^\star$. One simple example of such a compact family was already mentioned in the Introduction section.

If the composition of mappings is the semigroup operation, then because of Proposition 1, any nonempty compact subfamily $\mathscr{F}$ of $\mathscr{P}_d^\star$ generates a topological semigroup denoted by $\langle\mathscr{F}\rangle$, which in turn can naturally be associated with a Julia-type set. The primary objective of this article is to investigate such Julia sets and, more specifically, the approximation of such sets. The reason for invoking the concept of a semigroup in this context will be explained at the end of Section 5.

## 3. The Space $\mathscr{R}$ of Pluriregular Sets

If $E$ is a nonempty compact subset of $\mathbb{C}^N$, its pluricomplex Green function will be denoted by $V_E$. For the background, we refer the reader to [9]. Recall that

$$V_E=\log\Phi_E, \tag{9}$$

where $\Phi_E$ is the *Siciak extremal function*

$$\Phi_E(z)=\sup_p\left\{|p(z)|^{1/\deg p}\right\},\quad z\in\mathbb{C}^N, \tag{10}$$

with the supremum being taken over all nonconstant complex polynomials $p:\mathbb{C}^N\to\mathbb{C}$ such that $\|p\|_E\leq 1$. It is easy to check that for any compact set $E$, the zero set of $V_E$ is equal to the polynomially convex hull of $E$. A compact set $E$ is said to be *pluriregular* if $V_E$ is continuous.

Let $\mathscr{R}$ be the family of all compact, pluriregular, and polynomially convex subsets of $\mathbb{C}^N$. Endowed with metric $\Gamma$ defined by

$$\begin{aligned}\Gamma(E,F)&=\max\left\{\|V_E\|_F,\|V_F\|_E\right\}\\ &=\|V_E-V_F\|_{\mathbb{C}^N},\quad E,F\in\mathscr{R},\end{aligned} \tag{11}$$

$\mathscr{R}$ turns out to be a complete metric space (see Theorem 1 in [6]). It is worth observing that the above formula defining $\Gamma(E,F)$ can also be used for pluriregular sets $E$ and $F$ which are not necessarily polynomially convex. In this case, we obtain a pseudometric on the set of all pluriregular compact subsets of $\mathbb{C}^N$. Note also that if $E,F\in\mathscr{R}$, and $C$ is a set such that $E\cup F\subset C$, then $\Gamma(E,F)=\|V_E-V_F\|_C$.

It was shown in Theorem 1(a) from [1] that if $\mathscr{K}$ is compact in $\mathscr{R}$, then

$$\bigcup\mathscr{K}=\bigcup_{K\in\mathscr{K}}K\subset\mathbb{C}^N \tag{12}$$

is compact, being bounded and closed. In contrast, according to Theorem 1(d) in [1], a closed and bounded set in $\mathscr{R}$ does not need to be compact, since the space is not proper. In connection with these results, we would like to address here two questions, the answers to which can facilitate a better understanding of the topology of space $\mathscr{R}$.

The first question concerns the operations of closure in $\mathscr{R}$ and in $\mathbb{C}^N$. Let $\mathscr{G} \subset \mathscr{R}$. Is it true that

$$\bigcup \bar{\mathscr{G}} = \overline{\bigcup \mathscr{G}}? \tag{13}$$

It turns out that the answer depends on these additional assumptions:

(i) It is affirmative, if $\mathscr{G}$ is compact in $\mathscr{R}$. Indeed, $\bigcup \bar{\mathscr{G}} = \bigcup \mathscr{G} = \overline{\bigcup \mathscr{G}}$, where the second equality follows from Theorem 1(a) in [1].

(ii) However, the equality (13) is not true in the general case. To be more precise, we have the following properties:

(1) If $\mathscr{G}$ is relatively compact in $\mathscr{R}$, the inclusion "$\supset$" in (13) holds. Namely, $\bigcup \bar{\mathscr{G}}$ is closed by Theorem 1(a) in [1] and the inclusion follows from

$$\bigcup \bar{\mathscr{G}} \supset \bigcup \mathscr{G}. \tag{14}$$

(2) The inclusion "$\subset$" in (13) does not hold in general, even for a relatively compact set $\mathscr{G}$. To see this, consider the following example from Section 3 in [6]. Take $E_j = \{e^{it} : t \in [0, 2\pi - j^{-1}]\}$ and $\mathscr{G} = \{E_j : j \in \{1, 2, \dots\}\}$. We have

$$\begin{aligned} \bigcup \bar{\mathscr{G}} &= \{z \in \mathbb{C} : |z| \le 1\}, \\ \overline{\bigcup \mathscr{G}} &= \{z \in \mathbb{C} : |z| = 1\}. \end{aligned} \tag{15}$$

(3) If $\mathscr{G}$ is not relatively compact, the inclusion "$\supset$" in (13) does not need to hold either. To see this, recall Example 3.6 from [13]. Take $\mathscr{G} = \{E_n : n \in \{1, 2, \dots\}\}$ with

$$E_n := [1, 2] \cup \bigcup_{j=0}^{n-1} \left[\frac{j}{n}, \frac{j}{n} + \varepsilon_n\right], \tag{16}$$

where $\varepsilon_n > 0$ is so small that

$$\operatorname{cap}(E_n) \le \operatorname{cap}([1, 2]) + 1/n, \tag{17}$$

with $\operatorname{cap}(\cdot)$ denoting the logarithmic capacity. There exists $a \in [0, 1/2]$ such that

$$\lim_{n\to\infty} V_{E_n}(a) = V_{[1,2]}(a) > 0. \tag{18}$$

On the other hand, if $x \in \bigcup \bar{\mathscr{G}}$, then there exists $K \in \bar{\mathscr{G}}$ with $x \in K$, which means that we can find a subsequence $(E_{n_k})$ such that $\Gamma(E_{n_k}, K) \to 0$ as $k \to \infty$. At the same time,

$$0 \le V_{E_{n_k}}(x) \le \left\| V_{E_{n_k}} \right\|_K \le \Gamma(E_{n_k}, K). \tag{19}$$

Therefore, in this case, $V_{E_{n_k}}(x) \to 0$ as $k \to \infty$. Thus, $a \notin \bigcup \bar{\mathscr{G}}$. Hence, $\bigcup \bar{\mathscr{G}} \not\supset [0, 2] = \overline{\bigcup \mathscr{G}}$.

The other question concerns the fact that in $\mathbb{C}^N$ the compactness of a subset is equivalent to being closed and bounded, but it is not the case in $\mathscr{R}$. It is natural to ask whether compactness is needed in the assumption of Theorem 1(a) in [1] mentioned earlier. Let $\mathscr{K}$ be closed and bounded in $\mathscr{R}$. Does $\bigcup \mathscr{K}$ have to be compact in $\mathbb{C}^N$? The answer is no, it does not. Take $\mathscr{K} = \bar{\mathscr{G}}$ and $a \in [0, 1/2]$ from point (3) above (we use once again Example 3.6 in [13]). Since $a \in \overline{\bigcup \mathscr{G}}$, there exists a sequence $(a_n)_{n=1}^{\infty} \subset \bigcup \mathscr{G}$ with $a_n \to a$. Since $a \notin \bigcup \mathscr{K}$ and $(a_n) \subset \bigcup \mathscr{K}$, the set $\bigcup \mathscr{K}$ is not closed.

## 4. Similitudes of the Space of Pluriregular Sets

Let us recall the transformation formula for regular polynomial mappings from Theorem 5.3.1 in [9]:

$$V_{P^{-1}(E)} = \frac{1}{d} V_E \circ P, \quad E \subset \mathbb{C}^N, P \in \mathscr{P}_d^{\star}. \tag{20}$$

Recall also that if $(X, \mathfrak{d})$ is a metric space and $c > 0$ is a constant, then a mapping $f : X \to X$ is referred to as a *similitude with the ratio c*, if $\mathfrak{d}(f(a), f(b)) = c\,\mathfrak{d}(a, b)$ for all $a, b \in X$. As a direct consequence of (20), we can describe a family of similitudes of $\mathscr{R}$.

**Proposition 2.** *If $P \in \mathscr{P}_d^{\star}$, then*

$$A_P : \mathscr{R} \ni K \mapsto P^{-1}(K) \in \mathscr{R} \tag{21}$$

*is a contractive similitude with the contraction ratio $1/d$.*

*Proof 2.* Let $K, L \in \mathscr{R}$. In view of (20) we have

$$\begin{aligned} \Gamma\left(P^{-1}(K), P^{-1}(L)\right) &= \left\| V_{P^{-1}(K)} - V_{P^{-1}(L)} \right\|_{\mathbb{C}^N} \\ &= \frac{1}{d} \left\| V_K - V_L \right\|_{P(\mathbb{C}^N)} \\ &= \frac{1}{d} \Gamma(K, L). \end{aligned} \tag{22}$$

And this concludes the proof.

In particular, $A_P$ is a continuous map. Moreover, for each $R > 0$, the mapping

$$\mathscr{P}_d^{\star} \ni P \mapsto P^{-1}(\mathbb{B}_R) \in \mathscr{R} \tag{23}$$

is continuous (see Remark 1 in [8]). These observations can be generalized as follows.

**Proposition 3.** *The mapping*

$$\mathscr{P}_d^{\star} \times \mathscr{R} \ni (P, K) \mapsto P^{-1}(K) \in \mathscr{R} \tag{24}$$

*is continuous with respect to the product topology on $\mathscr{P}_d^{\star} \times \mathscr{R}$.*

*Proof 3.* Fix $K \in \mathscr{R}$ and $Q \in \mathscr{P}_d^{\star}$. In view of the triangle inequality and Proposition 2, if $P \in \mathscr{P}_d^{\star}$, $E \in \mathscr{R}$, then

$$\begin{aligned} \Gamma\left(P^{-1}(E), Q^{-1}(K)\right) &\le \Gamma\left(P^{-1}(E), P^{-1}(K)\right) + \Gamma\left(P^{-1}(K), Q^{-1}(K)\right) \\ &= \frac{1}{d} \Gamma(E, K) + \Gamma\left(P^{-1}(K), Q^{-1}(K)\right). \end{aligned} \tag{25}$$

Hence, it is now enough to prove that if $Q_n \to Q$, as $n \to \infty$, then

$$\Gamma\left(Q_n^{-1}(K), Q^{-1}(K)\right) \longrightarrow 0. \tag{26}$$

Take a sequence $(Q_n)_{n=1}^{\infty} \subset \mathscr{P}_d^{\star}$ which is convergent to $Q$ and consider $\mathscr{F} := \{Q_n : n \in \mathbb{N}\} \cup \{Q\}$. This family is compact in $\mathscr{P}_d^{\star}$ and therefore, $\bigcup_{P\in\mathscr{F}} P^{-1}(K)$ is bounded in view of Remark 3.2 in [12], because $K \in \mathscr{R}$. Take $\rho > 0$ such that $\bigcup_{P\in\mathscr{F}} P^{-1}(K) \subset \mathbb{B}_\rho$. Since $\|\cdot\|_{\mathbb{B}_\rho}$ is a norm in $\mathscr{P}_d$, there exists $m > 0$ such that $\|Q_n\|_{\mathbb{B}_\rho} \le R := \|Q\|_{\mathbb{B}_\rho} + 1$ for $n \ge m$. This means that $Q_n(\mathbb{B}_\rho) \subset \mathbb{B}_R$ for such $n$. Obviously, $Q(\mathbb{B}_\rho) \subset \mathbb{B}_R$, too.

Fix $\varepsilon > 0$. The Green function $V_K$ is continuous; hence, it is uniformly continuous on $\mathbb{B}_R$, that is, there exists $\delta > 0$ such that if $z, w \in \mathbb{B}_R$ with $\|z - w\| < \delta$, then $|V_K(z) - V_K(w)| < \varepsilon$. Since $Q_n \to Q$, there exists $k \ge m$ such that $\|Q_n - Q\|_{\mathbb{B}_\rho} < \delta$ if $n \ge k$. Therefore,

$$\Gamma\left(Q_n^{-1}(K), Q^{-1}(K)\right) = \frac{1}{d}\|V_K \circ Q_n - V_K \circ Q\|_{\mathbb{B}_\rho} < \varepsilon \quad \text{if } n \ge k. \tag{27}$$

And this concludes the proof.

Let $\mathscr{F}$ now be a compact subset of $\mathscr{P}_d^{\star}$. For any subset $\mathscr{K}$ of $\mathscr{R}$, put

$$\mathscr{A}_{\mathscr{F}}(\mathscr{K}) := \bigcup_{P\in\mathscr{F}} A_P(\mathscr{K}), \tag{28}$$

where the similitudes $A_P$ are as in Proposition 2.

**Proposition 4.** *If $\mathscr{F}$ is a compact subset of $\mathscr{P}_d^{\star}$ and $\mathscr{K}$ is a compact subset of $\mathscr{R}$, then $\mathscr{A}_{\mathscr{F}}(\mathscr{K})$ is compact.*

*Proof 4.* Choose a sequence $(E_n)$ of elements from $\mathscr{A}_{\mathscr{F}}(\mathscr{K})$. Then, there exist sequences $(P_n) \subset \mathscr{F}$ and $(K_n) \subset \mathscr{K}$ such that $E_n = P_n^{-1}(K_n)$. As $\mathscr{K}$ is compact, we can assume (passing to a subsequence if needed) that $K_n \to K$ in $\mathscr{K}$ if $n \to \infty$. Since $\mathscr{F}$ is compact, so here again (passing to a subsequence if needed), we can assume that $P_n \to P$ in $\mathscr{F}$ if $n \to \infty$. It follows from Proposition 3 that $P_n^{-1}(K_n) \to P^{-1}(K)$, if $n \to \infty$. Thus, we have shown that every sequence in $\mathscr{A}_{\mathscr{F}}(\mathscr{K})$ has a convergent subsequence.

Recall that $\kappa(X)$ denotes the family of all nonempty compact subsets of the metric space $X$, furnished with the Hausdorff metric.

**Corollary 1.** *Let $\mathscr{F}$ be a nonempty compact subset of $\mathscr{P}_d^{\star}$. The mapping*

$$\mathscr{A}_{\mathscr{F}} : \kappa(\mathscr{R}) \ni \mathscr{K} \mapsto \bigcup_{P\in\mathscr{F}} A_P(\mathscr{K}) \in \kappa(\mathscr{R}) \tag{29}$$

*is well defined and is a contraction with ratio $1/d$. In particular, the mapping $\mathscr{A}_{\mathscr{F}}$ has a unique fixed point $\mathcal{S}[\mathscr{F}] \in \kappa(\mathscr{R})$.*

*Proof 5.* It is enough to use the inequality

$$\Gamma\left(\bigcup_{j\in J} E_j, \bigcup_{j\in J} F_j\right) \le \sup_{j\in J} \Gamma\left(E_j, F_j\right), \quad \left(E_j\right)_{j\in J}, \left(F_j\right)_{j\in J} \subset \mathscr{R}, \tag{30}$$

(cf. [12], p. 891, and Corollary 2 in [6]) in combination with Propositions 2 and 4. The second conclusion follows from Banach's contraction principle.

## 5. Julia-Type Sets

If $P \in \mathscr{P}_d^{\star}$, its *(autonomous) filled-in Julia set* is defined as follows:

$$J[P] = \left\{z \in \mathbb{C}^N : \left(P^n(z)\right)_{n=1}^{\infty} \text{ is bounded}\right\}. \tag{31}$$

As shown in [6], this set is the unique fixed point of the similitude $A_P : \mathscr{R} \ni K \mapsto P^{-1}(K) \in \mathscr{R}$. Hence, the standard argument used to prove the Banach contraction principle yields the equality

$$J[P] = \lim_{n\to\infty} \left(P^n\right)^{-1}(E), \quad E \in \mathscr{R}. \tag{32}$$

Moreover, if $R > 0$ is an escape radius of $P$, then we also have the equality

$$J[P] = \bigcap_{n\ge 1} \left(P^n\right)^{-1}\left(\mathbb{B}_R\right). \tag{33}$$

Before turning our attention to other types of Julia sets, we need to point some useful estimates. If $R > 0$ is an escape radius for $P \in \mathscr{P}_d^{\star}$, then (cf. Equation 7 in [1])

$$\Gamma\left(P^{-1}\left(\mathbb{B}_R\right), \mathbb{B}_R\right) \le \frac{\|P\|_{\partial\mathbb{B}_R}}{Rd}. \tag{34}$$

More generally, if $\mathscr{F}$ is a compact family in $\mathscr{P}_d^{\star}$, then due to the continuity of the mapping in (8), a common escape radius $R > 0$ for all mappings in $\mathscr{F}$ can be found. Also,

$$M := \sup_{P\in\mathscr{F}} \|P\|_{\partial\mathbb{B}_R} \tag{35}$$

is finite because of the compactness of $\mathscr{F}$. Thus, as an immediate consequence of (34) we obtain

$$\Gamma\left(P^{-1}\left(\mathbb{B}_R\right), \mathbb{B}_R\right) \le \frac{M}{Rd}, \quad P \in \mathscr{F}. \tag{36}$$

For a sequence $(P_n)_{n=1}^{\infty}$ of mappings from $\mathscr{F}$, we define its *filled-in Julia set* (*nonautonomous* if the sequence is not constant) as follows:

$$J\left[\left(P_n\right)_{n=1}^{\infty}\right] = \left\{z \in \mathbb{C}^N : \left(\left(P_n \circ \ldots \circ P_1\right)(z)\right)_{n=1}^{\infty} \text{ is bounded}\right\}. \tag{37}$$

The estimate (36) allows the use of the enhanced version of Banach's contraction principle (Lemma 4.5 in [12]) for sequence $(A_{P_n})_{n=1}^{\infty}$. As a consequence, we can see that

$$J\left[(P_n)_{n=1}^{\infty}\right] = \lim_{n\to\infty}(P_n\circ\cdots\circ P_1)^{-1}(E), \quad E\in\mathscr{R},$$
$$J\left[(P_n)_{n=1}^{\infty}\right] = \bigcap_{n\geq 1}(P_n\circ\cdots\circ P_1)^{-1}(\mathbb{B}_R), \tag{38}$$

if $R>0$ is as in (36). For some background on (a larger family of) nonautonomous Julia sets in the complex plane, see [14, 15].

It turns out that nonautonomous filled-in Julia sets can be approximated by autonomous filled-in Julia sets. Before making this statement more precise, let us establish some notations. If $\mathscr{F}$ is a compact family in $\mathscr{P}_d^{\star}$, the symbol $\mathscr{F}^{\mathbb{N}}$ will denote the *code space over* $\mathscr{F}$, defined as the Cartesian product of countably many copies of $\mathscr{F}$ with the usual product topology. By Tychonoff's theorem, $\mathscr{F}^{\mathbb{N}}$ is compact and it can be furnished with the metric (see, e.g., Theorem 4.2.2 in [16]):

$$\rho\left((P_n)_{n=1}^{\infty},(Q_n)_{n=1}^{\infty}\right) = \sum_{j=1}^{\infty}\frac{\left\|P_j-Q_j\right\|_{\mathbb{B}_R}}{2^j}, \quad (P_n)_{n=1}^{\infty},(Q_n)_{n=1}^{\infty}\in\mathscr{F}^{\mathbb{N}}. \tag{39}$$

**Proposition 5.** *Let $\mathscr{F}$ be a compact family in $\mathscr{P}_d^{\star}$. Then, for each $(P_n)_{n=1}^{\infty}\in\mathscr{F}^{\mathbb{N}}$ and $m\in\mathbb{N}$*

$$\Gamma\left(J\left[(P_n)_{n=1}^{\infty}\right],(P_m\circ\cdots\circ P_1)^{-1}(\mathbb{B}_R)\right)\leq\frac{M}{Rd^m(d-1)}, \tag{40}$$

*where $M := \sup_{P\in\mathscr{F}}\|P\|_{\partial\mathbb{B}_R}$. In particular,*

$$J\left[(P_n)_{n=1}^{\infty}\right] = \lim_{m\to\infty}J[P_m\circ\cdots\circ P_1]. \tag{41}$$

*Proof 6.* To show (40) one can repeat the proof of the enhanced version of Banach's contraction principle (Lemma 4.5 in [12]). Namely, in view of (36), we have

$$\begin{aligned}\Gamma&\left((P_{n+m}\circ\cdots\circ P_1)^{-1}(\mathbb{B}_R),(P_m\circ\cdots\circ P_1)^{-1}(\mathbb{B}_R)\right)\\&\leq\frac{M}{Rd}\sum_{j=1}^{n}\frac{1}{d^{m+j-1}} = \frac{M}{Rd^m}\sum_{j=1}^{n}\frac{1}{d^j}.\end{aligned} \tag{42}$$

Letting $n$ go to infinity gives (40).

As for (41), in view of (40) we can write

$$\begin{aligned}\Gamma&\left(J\left[(P_n)_{n=1}^{\infty}\right],J[P_m\circ\cdots\circ P_1]\right)\\&\leq\Gamma\left(J\left[(P_n)_{n=1}^{\infty}\right],(P_m\circ\cdots\circ P_1)^{-1}(\mathbb{B}_R)\right)\\&\quad+\Gamma\left((P_m\circ\cdots\circ P_1)^{-1}(\mathbb{B}_R),J[P_m\circ\cdots\circ P_1]\right)\\&\leq\frac{2M}{Rd^m(d-1)}.\end{aligned} \tag{43}$$

For a finite $\mathscr{F}$, Proposition 5 was shown in [17].

We define the *partly filled-in composite Julia set* of the compact family $\mathscr{F}\subset\mathscr{P}_d^{\star}$ as

$$\mathbb{J}_{\mathrm{tr}}[\mathscr{F}] = \bigcap_{m\in\mathbb{N}}\left[\bigcup_{P_1,\ldots,P_m\in\mathscr{F}}(P_m\circ\cdots\circ P_1)^{-1}(\mathbb{B}_R)\right]. \tag{44}$$

This set is compact (see proof of Theorem 4.6 in [12]), and its polynomially convex hull $\mathbb{J}[\mathscr{F}]$ is the unique fixed point of the mapping:

$$\mathscr{R}\ni K\mapsto\widehat{\bigcup_{P\in\mathscr{F}}A_P(K)}\in\mathscr{R}. \tag{45}$$

$\mathbb{J}[\mathscr{F}]$ is called the *filled-in composite Julia set* of $\mathscr{F}$. Here, the hat marks the operation of taking the polynomially convex hull of the set under the hat. The subscript tr stands for the word truncated.

The following theorem describes the connection between the Julia sets from this section and the attractor $\mathcal{S}[\mathscr{F}]$ from the end of the previous section (Corollary 1).

**Theorem 1.** *Let $\mathscr{F}$ be a nonempty compact family in $\mathscr{P}_d^{\star}$. Then,*

*(1) $\mathcal{S}[\mathscr{F}] = \{J[(P_n)_{n=1}^{\infty}]: (P_n)_{n=1}^{\infty}\in\mathscr{F}^{\mathbb{N}}\}$;*

*(2) $\mathbb{J}_{\mathrm{tr}}[\mathscr{F}] = \bigcup\mathcal{S}[\mathscr{F}]$.*

*Proof 7.* This fact can be deduced from general theory in [18] but we give here the proof in this special case to make our work consistent (cf. also [19] for the case of a finite family).

The family $\mathcal{S}=\mathcal{S}[\mathscr{F}]$ is the unique fixed point of $\mathscr{A}_{\mathscr{F}}:\kappa(\mathscr{R})\to\kappa(\mathscr{R})$ (cf. Corollary 1). Therefore,

$$\begin{aligned}\mathcal{S}&=\mathscr{A}_{\mathscr{F}}(\mathcal{S})=\bigcup_{P_1\in\mathscr{F}}A_{P_1}(\mathcal{S})=\bigcup_{P_1,P_2\in\mathscr{F}}A_{P_1}\left(A_{P_2}(\mathcal{S})\right)\\&=\cdots=\bigcup_{P_1,P_2,\ldots,P_n\in\mathscr{F}}\left(A_{P_1}\circ A_{P_2}\circ\cdots\circ A_{P_n}\right)(\mathcal{S}),\quad n\in\mathbb{N}.\end{aligned} \tag{46}$$

Since by Proposition 2 the function $A_{P_j}$ is a contraction,

$$\mathrm{diam}_{\Gamma}\left(\left(A_{P_1}\circ A_{P_2}\circ\cdots\circ A_{P_n}\right)(\mathcal{S})\right)\longrightarrow 0,\quad n\longrightarrow\infty. \tag{47}$$

The sequence $((A_{P_1}\circ A_{P_2}\circ\cdots\circ A_{P_n})(\mathcal{S}))_{n=1}^{\infty}$ is also decreasing with respect to inclusion. Therefore, its limit is a singleton, and by the definition of $J[(P_n)_{n=1}^{\infty}]$, we have the equality

$$\bigcap_n\left(A_{P_1}\circ A_{P_2}\circ\cdots\circ A_{P_n}\right)(\mathcal{S}) = \left\{J\left[(P_n)_{n=1}^{\infty}\right]\right\}. \tag{48}$$

Thus, $\mathcal{S}=\{J[(P_n)_{n=1}^{\infty}]: (P_n)_{n=1}^{\infty}\in\mathscr{F}^{\mathbb{N}}\}$.

From the definition of $\mathbb{J}_{\mathrm{tr}}[\mathscr{F}]$, it is obvious that $\bigcup\mathcal{S}[\mathscr{F}]\subset\mathbb{J}_{\mathrm{tr}}[\mathscr{F}]$. Let us fix a common escape radius $R>0$ for all $P\in\mathscr{F}$.

Now, take $z\in\mathbb{J}_{\mathrm{tr}}[\mathscr{F}]$. First, we claim that for any $n\in\mathbb{N}$, there exists $E_n$ contained in the $1/n$-dilation of $\mathcal{S}[\mathscr{F}]$ and such that $z \in E_n$. Indeed, given $n \in \mathbb{N}$, one can choose $m \in \mathbb{N}$ so that the inequality

$$\frac{M}{Rd^m(d-1)} < \frac{1}{n} \tag{49}$$

is satisfied with $M$ defined in Proposition 5. Since $z \in \mathbb{J}_{\mathrm{tr}}[\mathscr{F}]$, by the definition of the latter, there exist $P_1, \ldots, P_m \in \mathscr{F}$ such that $z \in (P_m \circ \cdots \circ P_1)^{-1}(\mathbb{B}_R)$. The inequalities (49) and (40) imply the estimate

$$\Gamma\left((P_m \circ \cdots \circ P_1)^{-1}(\mathbb{B}_R), J[P_m \circ \cdots \circ P_1]\right) < \frac{1}{n}, \tag{50}$$

and this means that $E_n = (P_m \circ \cdots \circ P_1)^{-1}(\mathbb{B}_R)$ fulfills our claim.

To finish the proof, we want to show that $z \in \bigcup \mathcal{S}[\mathscr{F}]$. Since $\mathcal{S}[\mathscr{F}]$ is compact, the sequence $(E_n)$ has an accumulation point $E \in \mathcal{S}[\mathscr{F}]$. But then, since convergence of sets in $(\mathscr{R}, \Gamma)$ means uniform convergence of the corresponding pluricomplex Green functions, we can conclude that $V_E(z) = 0$, which means that $z \in E \subset \bigcup \mathcal{S}[\mathscr{F}]$.

*Remark 1.* It is worth emphasizing that all of the types of Julia sets defined in this section correspond one way or another to sequences in the semigroup $\langle \mathscr{F} \rangle$. This is the reason why conceptually it is natural to see the set $\mathcal{S}[\mathscr{F}]$ not only as the attractor associated with the semigroup $\langle \mathscr{F} \rangle$ but also as a kind of atlas of all Julia sets associated with that semigroup. Indeed, this is exactly the meaning of Theorem 2 combined with the definition of $\mathbb{J}_{\mathrm{tr}}[\mathscr{F}]$.

## 6. On the Attracting Nature of $\mathcal{S}[\mathscr{F}]$

Recall that we use the symbol $B_\Gamma(E, r)$ to denote the open ball in $(\mathscr{R}, \Gamma)$ with center at $E \in \mathscr{R}$ and radius $r > 0$.

The next theorem is a counterpart of Theorems 2(a) and 2(b) in [1] in the case of infinite compact regular families of polynomial mappings.

**Theorem 2.** *Let $\mathscr{F}$ be a nonempty compact family in $\mathscr{P}_d^\star$.*

(a) *Let $(\pi_n)_{n=1}^\infty \subset \mathscr{F}$. If $E \in \mathscr{R}$ and $\mathscr{U} \supset \mathcal{S}[\mathscr{F}]$ is an open subset of $\mathscr{R}$, then almost all elements of the sequence*

$$\mathscr{E} = \left\{ \left(A_{\pi_n} \circ \cdots \circ A_{\pi_1}\right)(E) : n \geq 1 \right\} \tag{51}$$

*belong to $\mathscr{U}$. In particular, all accumulation points of this sequence are in $\mathcal{S}[\mathscr{F}]$ and so $\bar{\mathscr{E}}$ is compact in $\mathscr{R}$.*

(b) *Let $E \in \mathcal{S}[\mathscr{F}]$. For every neighbourhood $\mathscr{V}$ of $E$, there exists an open set $\mathscr{U} \supset \mathcal{S}[\mathscr{F}]$ and mappings $Q_1, \ldots, Q_m \in \mathscr{F}$, such that*

$$\left(A_{Q_m} \circ \cdots \circ A_{Q_1}\right)(\mathscr{U}) \subset \mathscr{V}. \tag{52}$$

*Moreover, $m$ can be made arbitrarily large.*

*Proof 8.* ($a$) Fix a common escape radius $R > 0$ for all $P \in \mathscr{F}$. Let $M$ be like in Proposition 5. Fix $P \in \mathscr{F}$. In view of the proof of (40) (but with $E$ replacing $\mathbb{B}_R$) combined with the triangle inequality and (36), we have the following estimates:

$$\begin{aligned}
&\operatorname{dist}_\Gamma\left((\pi_1 \circ \cdots \circ \pi_m)^{-1}(E), \mathcal{S}[\mathscr{F}]\right) \\
&\quad \leq \Gamma\left((\pi_1 \circ \cdots \circ \pi_m)^{-1}(E), J[\pi_1 \circ \cdots \circ \pi_m]\right) \\
&\quad \leq \frac{\sup\left\{\Gamma\left(E, P^{-1}(E)\right) : P \in \mathscr{F}\right\}}{d^{m-1}(d-1)} \\
&\quad \leq \sup_{P \in \mathscr{F}} \frac{\Gamma(E, \mathbb{B}_R) + \Gamma\left(\mathbb{B}_R, P^{-1}(\mathbb{B}_R)\right) + \Gamma\left(P^{-1}(\mathbb{B}_R), P^{-1}(E)\right)}{d^{m-1}(d-1)} \\
&\quad \leq \frac{(d+1)\Gamma(E, \mathbb{B}_R) + M/R}{d^m(d-1)} \longrightarrow 0 \quad \text{if } m \longrightarrow \infty,
\end{aligned} \tag{53}$$

which is what is needed, as $\operatorname{dist}_\Gamma(\mathscr{R} \setminus \mathscr{U}, \mathcal{S}[\mathscr{F}]) > 0$.

($b$) Take $E \in \mathcal{S}[\mathscr{F}]$ and $\varepsilon > 0$ such that $B_\Gamma(E, \varepsilon) \subset \mathscr{V}$. Fix $n \in \mathbb{N}$. Without loss of generality, we may suppose that $d^{-n}\operatorname{diam}_\Gamma(\mathcal{S}[\mathscr{F}]) < \varepsilon/4$. It follows from Theorem 1 that $E = J[(P_n)_{n=1}^\infty]$ for some $(P_n)_{n=1}^\infty \in \mathscr{F}^{\mathbb{N}}$. Moreover, $J[(P_n)_{n=1}^\infty] = \lim_{m\to\infty} J[P_m \circ \cdots \circ P_1]$ by (41). Therefore, we can choose $m > n$ such that

$$\Gamma(E, J[P_m \circ \cdots \circ P_1]) < \frac{\varepsilon}{4}. \tag{54}$$

Define $Q_j = P_{m+1-j}$ for $j \in \{1, \ldots, m\}$ and let

$$\mathscr{U} = \{F \in \mathscr{R} : \operatorname{dist}_\Gamma(F, \mathcal{S}[\mathscr{F}]) < \varepsilon\}. \tag{55}$$

If $F \in \mathscr{U}$, then there exists $G \in \mathcal{S}[\mathscr{F}]$ such that $\Gamma(F, G) < \varepsilon$. Therefore,

$$\Gamma\left((Q_1 \circ \cdots \circ Q_m)^{-1}(F), (Q_1 \circ \cdots \circ Q_m)^{-1}(G)\right) \leq d^{-m}\Gamma(F, G) < \frac{\varepsilon}{2}. \tag{56}$$

Moreover,

$$\begin{aligned}
&\Gamma\left(J[P_m \circ \cdots \circ P_1], (Q_1 \circ \cdots \circ Q_m)^{-1}(G)\right) \\
&\quad = \Gamma\left((P_m \circ \cdots \circ P_1)^{-1}(J[P_m \circ \cdots \circ P_1]), (P_m \circ \cdots \circ P_1)^{-1}(G)\right) \\
&\quad \leq d^{-m}\Gamma(J[P_m \circ \cdots \circ P_1], G) \leq d^{-m}\operatorname{diam}_\Gamma(\mathcal{S}[\mathscr{F}]) < \frac{\varepsilon}{4}.
\end{aligned} \tag{57}$$

Combining (54), (56), (57), and using the triangle inequality, we see that

$$\Gamma\left(E, (Q_1 \circ \cdots \circ Q_m)^{-1}(F)\right) < \varepsilon, \tag{58}$$

as required.

## 7. Chaos Game and Approximation of Attractors

We will start with the definition of disjunctive sequences over a finite or countable alphabet.

Let $A$ be a nonempty set which is at most countable. A sequence of elements of $A$, that is, a function $\tau : \mathbb{N} \to A$ is said to be *disjunctive*, if for any $m \in \mathbb{N}$ and any function $\theta : \{1, \ldots, m\} \to A$ there exists $n \in \mathbb{N}$

such that $\theta(j)=\tau(n+j)$ for $j\in\{1,\dots,m\}$. A simple example of a disjunctive sequence with $A=\mathbb{N}$ is given in [20]: the first entry is 1, followed by all 2-letter words over $\{1,2\}$, then by all 3-letter words over $\{1,2,3\}$, and so on.

If $A$ is regarded as the *alphabet* and functions like $\theta$ as possible *finite words over A*, then the sequence $\tau$ is disjunctive if it contains all finite words as its finite subsequences. Disjunctive sequences, usually over a finite alphabet, have been used for a long time in study of formal languages, in automata theory and number theory (see [21] for an overview). More recently, disjunctive sequences turned out to be a natural tool for derandomization of the chaos game (see [20]).

The next result is a generalization of Theorem 2(c) in [1].

**Theorem 3.** *Let $\mathscr{F}$ be a nonempty compact subset of $\mathscr{P}_d^\star$ and $\mathscr{F}_0=\{\pi_n : n\in\mathbb{N}\}$ a dense countable subset of $\mathscr{F}$. Let $\tau:\mathbb{N}\to\mathbb{N}$ be a disjunctive sequence.*

*Then, for any $E\in\mathscr{R}$,*

$$\lim_{m\to\infty}\Gamma\left(\mathbb{J}[\mathscr{F}],\ \bigcup\overline{\mathscr{E}_m}\right)=0, \tag{59}$$

*where*

$$\mathscr{E}_m=\left\{\left(\pi_{\tau(1)}\circ\cdots\circ\pi_{\tau(n)}\right)^{-1}(E)\colon n\geq m\right\}. \tag{60}$$

*Proof 9.* First of all, it should be noted that Theorem 2 yields compactness of $\overline{\mathscr{E}_m}$. Furthermore, a countable subset $\mathscr{F}_0$ of $\mathscr{F}$ exists because of the separability of $\mathscr{P}_d$. Recall also that $\mathscr{R}$ is separable (see Theorem 1(d) in [1]).

Fix a norm $\|\cdot\|$ in $\mathscr{P}_d$.

In view of Theorem 2(b) and from Theorem 1(b) in [1], it suffices to prove that

$$\lim_{m\to\infty}\chi_\Gamma\left(\mathscr{S}[\mathscr{F}],\overline{\mathscr{E}_m}\right)=0, \tag{61}$$

where $\chi_\Gamma$ denotes the Hausdorff metric corresponding to $\Gamma$.

Take $\varepsilon>0$. In view of Theorem 2(a), if $m$ is sufficiently large, then the $\varepsilon$-dilation of $\mathscr{S}[\mathscr{F}]$ contains $\mathscr{E}_m$, and hence also $\overline{\mathscr{E}_m}$. In order to prove that for sufficiently large $m$, the $\varepsilon$-dilation of $\mathscr{E}_m$ contains $\mathscr{S}[\mathscr{F}]$, it is enough to show that any point from an $\varepsilon/2$-dense finite subset of $\mathscr{S}[\mathscr{F}]$ is within $\varepsilon/2$-distance from a point of $\mathscr{E}_m$.

Let $A\in\mathscr{S}[\mathscr{F}]$ be an element of a fixed $\varepsilon/2$-dense finite subset of $\mathscr{S}[\mathscr{F}]$. By Theorem 2(b), there exist $\ell\in\mathbb{N}$ and $Q_1,\dots,Q_\ell\in\mathscr{F}$ such that for $\delta\in(0,\varepsilon/2)$ the image of the $\delta$-dilation of $\mathscr{S}[\mathscr{F}]$ via the mapping

$$F\mapsto\left(Q_1\circ\cdots\circ Q_\ell\right)^{-1}(F) \tag{62}$$

is a subset of $B_\Gamma(A,\varepsilon/4)$. Using Theorem 2(a) again if necessary, we can increase $m$ so that the $\delta$-dilation of $\mathscr{S}[\mathscr{F}]$ contains $\mathscr{E}_m$. In particular, if $K\in\mathscr{E}_m$, then

$$\left(Q_1\circ\cdots\circ Q_\ell\right)^{-1}(K)\in B_\Gamma\left(A,\frac{\varepsilon}{4}\right). \tag{63}$$

Proposition 3 assures continuity of $\mathscr{P}_{d^l}^\star\times\mathscr{R}\ni(Q,K)\mapsto Q^{-1}(K)\in\mathscr{R}$. By Proposition 1, the mapping $\mathscr{F}^l\ni(q_1,\dots,q_l)\mapsto q_1\circ\cdots\circ q_\ell\in\mathscr{P}_{d^l}^\star$ is continuous, too. Therefore, the mapping

$$\mathscr{F}^l\times\overline{\mathscr{E}_m}\ni(q_1,\dots,q_\ell,K)\mapsto(q_1\circ\cdots\circ q_\ell)^{-1}(K)\in\mathscr{R} \tag{64}$$

is uniformly continuous. Thus, there exists $\eta>0$ such that if $p_j,q_j\in\mathscr{F}$ with $\|q_j-p_j\|<\eta, j\in\{1,\dots,\ell\}$, and $K,L\in\overline{\mathscr{E}_m}$ with $\Gamma(K,L)<\eta$, then

$$\Gamma\left((q_1\circ\cdots\circ q_\ell)^{-1}(K),(p_1\circ\cdots\circ p_\ell)^{-1}(L)\right)<\frac{\varepsilon}{4}. \tag{65}$$

Since $\mathscr{F}_0$ is dense in $\mathscr{F}$, there exist $P_1,\dots,P_\ell\in\mathscr{F}_0$ such that $\|P_1-Q_1\|<\eta,\dots,\|P_\ell-Q_\ell\|<\eta$. Let $\theta:\{1,\dots,\ell\}\to\mathbb{N}$ be chosen so that $P_j=\pi_{\theta(j)}$ for $j\in\{1,\dots,\ell\}$.

Since $\tau$ is disjunctive, for some $n\geq m$, we have $\theta(j)=\tau(n+j)$ for $j\in\{1,\dots,\ell\}$. Consequently, if we put $K=(\pi_{\tau(1)}\circ\cdots\circ\pi_{\tau(n)})^{-1}(E)$, we have $K\in\mathscr{E}_m$ and

$$\begin{aligned}\left(\pi_{\tau(1)}\circ\cdots\circ\pi_{\tau(n+\ell)}\right)^{-1}(E)&=\left(\pi_{\theta(1)}\circ\cdots\circ\pi_{\theta(\ell)}\right)^{-1}(K)\\&=(P_1\circ\cdots\circ P_\ell)^{-1}(K).\end{aligned} \tag{66}$$

We know that

$$\Gamma\left((P_1\circ\cdots\circ P_\ell)^{-1}(K),(Q_1\circ\cdots\circ Q_\ell)^{-1}(K)\right)<\frac{\varepsilon}{4}, \tag{67}$$

because of the choice of $\eta$, and so it follows from (63) combined with the triangle inequality that

$$\left(\pi_{\tau(1)}\circ\cdots\circ\pi_{\tau(n+\ell)}\right)^{-1}(E)\in B_\Gamma\left(A,\frac{\varepsilon}{2}\right). \tag{68}$$

And this concludes the proof.

The next statement is a probabilistic version of the above theorem.

**Corollary 2.** *Let $\mathscr{F}$ be a nonempty compact subset of $\mathscr{P}_d^\star$ and $\mathscr{F}_0=\{\pi_n : n\in\mathbb{N}\}$ a dense countable subset of $\mathscr{F}$. Let $\tau:\mathbb{N}\to\mathbb{N}$ be generated according to probabilities $p_1,p_2,\dots>0$ such that $\sum_{n=1}^\infty p_n=1$, that is, the values $\tau(j)$ of $\tau$ are chosen at random, independent from each other, so that $\mathbb{P}[\tau(j)=i]=p_i$ for $i,j\in\mathbb{N}$.*

*Then, for any $E\in\mathscr{R}$, with probability 1,*

$$\lim_{m\to\infty}\Gamma\left(\mathbb{J}[\mathscr{F}],\ \bigcup\overline{\mathscr{E}_m}\right)=0, \tag{69}$$

*where $\mathscr{E}_m=\{(\pi_{\tau(1)}\circ\cdots\circ\pi_{\tau(n)})^{-1}(E)\colon n\geq m\}$.*

*Proof 10.* Because of the strong law of large numbers applied to Bernoulli processes, we can conclude that, given a finite word over the alphabet $\mathbb{N}$, the sequence $\tau$ contains this word with probability 1. Hence, we can use the same reasoning as in the theorem above.

We would like to finish the article with a general observation.

Let us assume that we have a probability measure $W$ on some $\sigma$-algebra of subsets of $\mathscr{F}$, where as in Theorem 3, $\mathscr{F}$ is a compact subset of $\mathscr{P}_d^\star$. We will follow the general set-up from [7]. We will be concerned with a Markov chain $Z_n^K$, with initial state $K \in \mathscr{R}$ and

$$Z_k^K\left[(P_n)_{n=1}^\infty\right] = \begin{cases} K & \text{if } k=0, \\ \left(A_{P_k} \circ \cdots \circ A_{P_1}\right)(K) & \text{if } k \geq 1, \end{cases} \tag{70}$$

where $P_k$ are independently and identically distributed (IID) random elements in $\mathscr{F}$ with probability distribution $W$. Let $W$ also denote the induced probability measure on the code space $\mathscr{F}^{\mathbb{N}}$.

In a more general setting, the initial state can be given by a random element $X_0$ in $\mathscr{R}$, independent of $(P_n)_{n=1}^\infty$, and with the probability distribution $\nu$. Then, it is natural to define the random elements

$$Z_k^\nu\left[(P_n)_{n=1}^\infty\right] = Z_k^{X_0}\left[(P_n)_{n=1}^\infty\right], \quad k \in \mathbb{Z}_+. \tag{71}$$

So in particular, $\nu$ is the probability distribution of $Z_0^\nu$. If we also define $F(\nu)$ as the probability distribution of $Z_1^\nu$, then the probability distribution of $Z_k^\nu$ is $F^k(\nu)$.

The reverse order chain is defined to be as follows:

$$\widehat{Z}_k^K\left[(P_n)_{n=1}^\infty\right] = \begin{cases} K & \text{if } k=0, \\ \left(A_{P_1} \circ \cdots \circ A_{P_k}\right)(K) & \text{if } k \geq 1. \end{cases} \tag{72}$$

Because of the IID property, both $Z_k^\nu$ and $\widehat{Z}_k^\nu$ have the same probability distribution $F^k(\nu)$.

Note that all of the above definitions make sense because Proposition 3 is guaranteeing appropriate measurability of the sets.

Let $\delta_a$ denote the Dirac measure concentrated at $a$, that is, $\delta_a(E) = \mathbf{1}_E(a)$. Below, we use the mapping

$$\Pi : \mathscr{F}^{\mathbb{N}} \ni (P_n)_{n=1}^\infty \mapsto J\left[(P_n)_{n=1}^\infty\right] \in \mathscr{S}[\mathscr{F}]. \tag{73}$$

It is continuous because of the estimate (40) combined with the definition (39) of the metric $\rho$ on $\mathscr{F}^{\mathbb{N}}$. Indeed, given $\varepsilon > 0$ and $Q = (Q_n)_{n=1}^\infty \in \mathscr{F}^{\mathbb{N}}$, choose $m$ so that $M/(Rd^m(d-1)) < \varepsilon/4$. If $P = (P_n)_{n=1}^\infty \in \mathscr{F}^{\mathbb{N}}$ is such that $\rho(P,Q) < \varepsilon/2^{m+1}$, then $\|P_m - Q_m\|_{\mathbb{B}_R} < \varepsilon/2$ and thus $\Gamma(\Pi(P), \Pi(Q)) < \varepsilon$ in view of (40) combined with the triangle inequality.

**Proposition 6.** *Let $\mathscr{F}$ be a nonempty compact subset of $\mathscr{P}_d^\star$, let $W$ be a probability measure on some $\sigma$-algebra of subsets of $\mathscr{F}$ and let $W$ also denote the induced probability measure on $\mathscr{F}^{\mathbb{N}}$.*

*Let $\mu$ be the pushforward measure on $\mathscr{R}$ obtained from the measure $W$ on the code space $\mathscr{F}^{\mathbb{N}}$ via the mapping $\Pi$. Then:*

*(a) If $\nu$ is a Borel probability measure, then $F^n(\nu) \to \mu$ weakly. In particular, $F(\mu) = \mu$ and $\mu$ is the unique probability measure invariant with respect to $F$.*

*(b) For all $K \in \mathscr{R}$ and for a.e. $(P_n)_{n=1}^\infty \subset \mathscr{F}$*

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{Z_k^K\left[(P_n)_{n=1}^\infty\right]} \longrightarrow \mu \tag{74}$$

*weakly.*

*(c) The support of $\mu$ is $\mathscr{S}[\mathscr{F}]$; hence, this is the unique fixed point of the iterated function system $\{A_P : P \in \mathscr{F}\}$. In particular, the support of $\mu$ is compact.*

*Proof 11.* (a) and (b) are straightforward consequences of [7] (Theorem 8).

(c) By Theorem 8 (15) from [7], there exists $n_0$, which may depend on $(P_n)_{n=1}^\infty$ and $\varepsilon$, such that

$$(P_n \circ \cdots \circ P_1)^{-1}(\mathbb{B}_R) \in \operatorname{supp}(\mu)^\varepsilon, \quad \text{if } n \geq n_0. \tag{75}$$

Therefore, $\mathscr{S}[\mathscr{F}] \subset \operatorname{supp}(\mu)$. On the other hand, $\Pi(\mathscr{F}^{\mathbb{N}}) = \mathscr{S}[\mathscr{F}]$ and hence $\operatorname{supp}(\mu) \subset \mathscr{S}[\mathscr{F}]$.

It should be noted that the novel element in the above observation is the compactness of the support of the measure in the case of infinite family and its invariance under the IFS in this case. Theorem 8 in [7] gives this property, but only in the case of finite iterated function systems.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## References

[1] A. Alghamdi and M. Klimek, "Probabilistic approximation of partly filled-in composite Julia sets," *Annales Polonici Mathematici*, vol. 119, no. 3, pp. 203–220, 2017.

[2] A. Hinkkanen and G. J. Martin, "The dynamics of semigroups of rational functions. I," *Proceedings of the London Mathematical Society*, vol. s3-73, no. 2, pp. 358–384, 1996.

[3] R. Stankewitz and H. Sumi, “Backward iteration algorithms for Julia sets of Möbius semigroups,” *Discrete and Continuous Dynamical Systems*, vol. 36, no. 11, pp. 6475–6485, 2016.

[4] H. Sumi, “Random complex dynamics and semigroups of holomorphic maps,” *Proceedings of the London Mathematical Society*, vol. 102, no. 1, pp. 50–112, 2011.

[5] H. Sumi, “Random complex dynamics and devil’s coliseums,” *Nonlinearity*, vol. 28, no. 4, pp. 1135–1161, 2015.

[6] M. Klimek, “Metrics associated with extremal plurisubharmonic functions,” *Proceedings of American Mathematical Society*, vol. 123, no. 9, pp. 2763–2770, 1995.

[7] M. F. Barnsley, J. E. Hutchinson, and Ö. Stenflo, “V-variable fractals: fractals with partial self similarity,” *Advances in Mathematics*, vol. 218, no. 6, pp. 2051–2088, 2008.

[8] M. Klimek, “On perturbations of pluriregular sets generated by sequences of polynomial maps,” *Annales Polonici Mathematici*, vol. 80, pp. 171–184, 2003.

[9] M. Klimek, “Pluripotential theory,” in *London Mathematical Society Monographs. New Series, 6. Oxford Science Publications*, The Clarendon Press, Oxford University Press, New York, 1991.

[10] W. Rudin, “Inverse iteration systems in $C^n$. Reprint of the 1980 edition,” in *Classics in Mathematics*, Springer, Berlin, 2008.

[11] M. Klimek, “Iteration of analytic multifunctions,” *Nagoya Mathematical Journal*, vol. 162, pp. 19–40, 2001.

[12] M. Klimek and M. Kosek, “Composite Julia sets generated by infinite polynomial arrays,” *Bulletin des Sciences Mathematiques*, vol. 127, no. 10, pp. 885–897, 2003.

[13] J. Siciak, “On metrics associated with extremal plurisubharmonic functions,” *Bulletin of the Polish Academy of Sciences-Mathematics*, vol. 45, no. 2, pp. 151–160, 1997.

[14] R. Brück and M. Büger, “Generalized iteration,” *Computational Methods and Function Theory*, vol. 3, no. 1, pp. 201–252, 2003.

[15] J. E. Fornaess and N. Sibony, “Random iterations of rational functions,” *Ergodic Theory and Dynamical Systems*, vol. 11, no. 4, pp. 1289–1297, 1991.

[16] R. Engelking, *General Topology*, PWN-Polish Scientific Publishers, Warsaw, 1977.

[17] M. Klimek, “Inverse iteration systems in $C^n$,” *Acta Universitatis Upsaliensis, Scr. Uppsala Univ. C Organ. Hist*, vol. 64, pp. 206–214, 1999.

[18] K. R. Wicks, *Fractals and Hyperspaces, Lecture Notes in Mathematics, 1492*, Springer-Verlag, Berlin, 1991.

[19] J. E. Hutchinson, “Fractals and self similarity,” *Indiana University Mathematics Journal*, vol. 30, no. 5, pp. 713–747, 1981.

[20] K. Leśniak, “Random iteration for infinite nonexpansive iterated function systems,” *Chaos*, vol. 25, no. 8, article 083117, 2015.

[21] C. S. Calude, L. Priese, and L. Staiger, *Disjunctive Sequences: An Overview. Centre for Discrete Mathematics and Theoretical Computer Science, Research Report Series, No. 063*, University of Auckland, 1997.